Nuclear spin dynamics and Zeno effect in quantum dots and defect centers

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We analyze nuclear-spin dynamics in quantum dots and defect centers with a bound electron under electron-mediated coupling between nuclear spins due to the hyperfine interaction ("J coupling" in NMR). Our analysis shows that the Overhauser field generated by the nuclei at the position of the electron has short-time dynamics quadratic in time for an initial nuclear-spin state without transverse coherence. The quadratic short-time behavior allows for an extension of the Overhauser-field lifetime through a sequence of projective measurements (quantum Zeno effect). We analyze the requirements on the repetition rate of measurements and the measurement accuracy to achieve such an effect. Further, we calculate the long-time behavior of the Overhauser field for effective electron Zeeman splittings larger than the hyperfine coupling strength and find, both in a Dyson-series expansion and a generalized master-equation approach, that for a nuclear-spin system with a sufficiently smooth polarization the electron-mediated interaction alone leads only to a partial decay of the Overhauser field by an amount of the order of the inverse number of nuclear spins interacting with the electron.

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I. INTRODUCTION

Technological advancements have made it possible to confine very few electrons in a variety of nanostructures such as nanowires, quantum dots, donor impurities, or defect centers.[1–22] One driving force behind these achievements is a series of proposals for using the spin of an electron as a qubit for quantum computing.[23–25] This spin interacts with the nuclear spins in the host material via the hyperfine interaction. While this interaction leads to decoherence of the electron-spin state, on one hand, it also provides an opportunity to create a local effective magnetic field (Overhauser field) for the electron by inducing polarization in the nuclear-spin system, which could be used, e.g., for rapid single-spin rotations.[26] Polarizing the nuclear-spin system is also one possible way to suppress hyperfine-induced decoherence[27,28] or it can be used as a source of spin polarization to generate a spin-polarized current. In any case, controlling the dynamics of the Overhauser field and, in particular, to prevent its decay, is thus of vital importance in the context of spintronics and quantum computation.[29]

In GaAs quantum dots the Overhauser field can become as large as 5 T. The build up, decay, and correlation time of the Overhauser field have been studied in a number of systems,[30–45] suggesting time scales for the decay on the order of seconds, minutes, or, in one case, even hours.[46]

In this paper we address the question: How can a large Overhauser field be preserved? That is, how can the Overhauser-field decay be suppressed or even prevented. The dynamics of the Overhauser field is governed by the mutual interaction between the nuclear spins. There is on one hand the direct dipolar coupling between the nuclear spins. On the other hand, due to the presence of a confined electron, there is also an indirect interaction: The coupling of the nuclear spins to the electron via the hyperfine interaction leads to an effective interaction between the nuclear spins that is known as the electron-mediated interaction. While the effect of this electron-mediated interaction on the decoherence of the *electron* has been studied previously,[47–49] theoretical studies of the decay of the Overhauser field have so far studied direct dipole-dipole interaction, and the effect of the hyperfine interaction was taken into account through the Knight shift that the electron induces via the hyperfine interaction.[50] In this paper we investigate the effect of the electron-mediated interaction between nuclear spins on the dynamics of the Overhauser field. While the direct dipolar coupling is always present, it can be weaker than the electron-mediated interaction for magnetic fields that are not too large and may be further reduced via NMR pulse sequences or by diluting the concentration of nuclear spins.[51] We find in our calculation that for effective electron Zeeman splittings ω (sum of Zeeman splittings due to the external magnetic and Overhauser fields) larger than the hyperfine coupling strength A, the decay of the Overhauser field due to the electron-mediated interaction is incomplete, i.e., that only a small fraction of the Overhauser-field decays. In a short-time expansion that is valid for ω larger than $A/\sqrt{N}$, where N is the number of nuclear spins with which the electron interacts, we find a quadratic initial decay on a time scale ($\hbar=1$) $\tau_e=N^{3/2}\omega/A^2$. We show that by performing repeated projective measurements on the Overhauser field, a quantum Zeno effect occurs, which allows one to preserve the Overhauser field even for relatively small effective electron Zeeman splittings larger than $A/\sqrt{N}$. Overall, we thus obtain the following picture: The Overhauser field can be preserved either by applying a large external magnetic field (it only decays by a small fraction for $\omega \gg A$) or by performing repeated projective measurements on the Overhauser field (with our calculation for the short-time dynamics being valid for $\omega \gg A/\sqrt{N}$).

In Sec. II we briefly review the quantum Zeno effect and give the corresponding main results for the case of the Overhauser field. We start our detailed discussion in Sec. III by writing down the Hamiltonian for the hyperfine interaction and by deriving an effective Hamiltonian for the electron-mediated interaction. In Sec. IV we derive an expression for the short-time behavior of the Overhauser-field mean value. In Secs. V and VI we address the long-time decay of the Overhauser field due to the electron-mediated interaction. Some technical details are deferred to Appendixes A and B.

II. ZENO EFFECT

The suppression of the decay of a quantum state due to frequently repeated measurements is known as the quantum Zeno effect. The concept of the quantum Zeno effect[52] is almost as old as quantum mechanics[53,54] and it remains one of the most intriguing quantum effects. It has been studied intensively from the theoretical side[55–57] and also experimental evidence has been found in recent years.[58]

Let us consider a two-level system initialized to the excited state $|e\rangle$ possibly coupled to an environment initially in state ρ_E so that the composite system is initialized to $\rho(0) = |e\rangle\langle e| \otimes \rho_E$. The state of the two-level system evolves and may decay to the ground state under the action of a Hamiltonian H for time $t>0$. At short times, the survival probability P_s in the excited state can be expanded in powers of the elapsed time t: $P_s(t)=1+\langle P_s\rangle_1 t+\langle P_s\rangle_2 t^2/2+\ldots$. For the given initial conditions, the t-linear term is necessarily zero: $\langle P_s\rangle_1 = -i\,\mathrm{Tr}_E\langle e|[H,\rho(0)]|e\rangle = 0$. If $\langle P_s\rangle_2 \neq 0$, the evolution of P_s at sufficiently short times is given by $P_s(t)=1-c_s t^2/\tau_s^2$ with the constant c_s and the time scale τ_s being system dependent. A projective measurement at time τ_m resets the system to the excited state with probability $P_s(\tau_m)$. Repeating the measurement m times at intervals $\tau_m \ll \tau_s$, the survival probability is $P_{s,\mathrm{meas}}(m\tau_m)=(1-c_s\tau_m^2/\tau_s^2)^m \approx 1-c_s m\tau_m/(\tau_s^2/\tau_m)$ for $c_s m\tau_m^2/\tau_s^2 \ll 1$. The survival probability at time $t=m\tau_m$ is thus increased due to the frequently repeated measurements; instead of a quadratic decay on a time scale τ_s without measurements, we have a linear decay on a time scale τ_s^2/τ_m.

A more complex observable such as the mean of the Overhauser-field z component $\langle h_z(t)\rangle = \mathrm{Tr}\{h_z\rho(t)\}$ [see Eq. (6)] may also show a Zeno effect. Whether $\langle h_z(t)\rangle$ shows an initial quadratic decay is, however, not obvious and actually depends on the initial state of the nuclear-spin system $\rho_I(0)$ (see the first paragraph of Sec. IV below for details). For the short-time behavior of $\langle h_z(t)\rangle$, we expand in a Taylor series,

$$\langle h_z(t)\rangle = \langle h_z(0)\rangle + t\langle h_z\rangle_1 + \frac{t^2}{2}\langle h_z\rangle_2 + \ldots, \tag{1}$$

with $\langle h_z\rangle_n = d^n\langle h_z(t)\rangle/dt^n|_{t=0}$. If $\langle h_z\rangle_1=0$, the t-linear term vanishes and the initial decay is quadratic in time. In Sec. IV we find that $\langle h_z\rangle_1=0$ under the condition that the initial nuclear-spin density matrix is diagonal in a basis of h_z eigenstates. In this case the initial decay is of the form

$$\frac{\langle h_z(t)\rangle}{\langle h_z(0)\rangle} = 1 - c\frac{t^2}{\tau_e^2}. \tag{2}$$

The time scale τ_e and the constant c are given below in Eqs. (17) and (18), respectively.

Let us now consider a sequence of repeated measurements of the Overhauser field $h_z(t)$. In the context of quantum dots, several proposals to implement such measurements have been put forward. These proposals take advantage of optically active dots,[59] gated double quantum dots in the spin-blockade regime,[60] or phase-estimation methods[61] (see Appendix B for more details). A measurement of h_z shall be performed after a time τ_m. If this measurement is projective, i.e., if it sets all the off-diagonal elements of the density matrix in a basis of h_z eigenstates to zero (we discuss requirements on the accuracy of the measurement in Appendix B), the dynamics after τ_m again follows Eq. (2). Repeating the measurement at times $2\tau_m, 3\tau_m, \ldots$ leads to a change in the decay of the Overhauser field in the same way as we described it for the two-level system above,

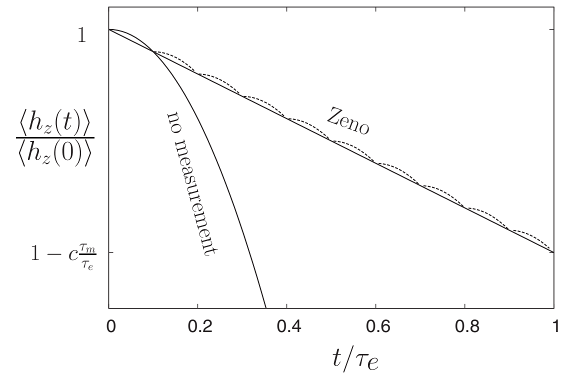

FIG. 1. Effect of projective measurements at time intervals $\tau_m=\tau_e/10$ on the time evolution of the Overhauser-field expectation value $\langle h_z(t)\rangle$. Due to the Zeno effect, the decay with measurements is $1-ct/\tau_{\mathrm{Zeno}}$ rather than $1-ct^2/\tau_e^2$ without measurements, where $\tau_{\mathrm{Zeno}}=\tau_e^2/\tau_m$. The formula $1-ct/\tau_{\mathrm{Zeno}}$ for the decay with measurement is only strictly valid at times $t=m\tau_m$ with m being a positive integer. After the measurement at $t=m\tau_m$ the decay is again quadratic with time dependence $\langle h_z(m\tau_m)\rangle/\langle h_z(0)\rangle - c(t-m\tau_m)^2/\tau_e^2$ (broken lines).

$$\frac{\langle h_z(t)\rangle_{\mathrm{Zeno}}}{\langle h_z(0)\rangle} = 1 - c\frac{t}{\tau_{\mathrm{Zeno}}}, \quad \tau_{\mathrm{Zeno}} = \frac{\tau_e^2}{\tau_m}. \tag{3}$$

Instead of a quadratic decay $\propto t^2/\tau_e^2$ we have a linear decay $\propto t/\tau_{\mathrm{Zeno}}$ with $\tau_{\mathrm{Zeno}}=\tau_e^2/\tau_m$. We note that the expression for $\langle h_z(t)\rangle_{\mathrm{Zeno}}$ in Eq. (3) is only strictly valid at times $m\tau_m$ with m being a positive integer. Between these times $\langle h_z(t)\rangle$ changes according to Eq. (2). The derivation of Eq. (3) requires $cm\tau_m^2/\tau_e^2 = ct/\tau_{\mathrm{Zeno}} \ll 1$. Figure 1 shows the Zeno effect, i.e., the difference between $\langle h_z(t)\rangle/\langle h_z(0)\rangle$ and $\langle h_z(t)\rangle_{\mathrm{Zeno}}/\langle h_z(0)\rangle$.

In addition to the requirements on the measurement accuracy (see Appendix B), the results in this section rest on the following separation of time scales:

$$\tau_{pm} \ll \tau_m \ll \tau_e, \tau_x, \tag{4}$$

where τ_{pm} is the time required to perform a single measurement and τ_x is the time scale up to which the short-time expansion for $\langle h_z(t)\rangle$ is valid. In general, τ_x can be shorter than τ_e. Further, the calculation for τ_e uses an effective Hamiltonian that neglects transfer of angular momentum from the electron to the nuclei and thus leads to an error in

$\langle h_z(t)\rangle/\langle h_z(0)\rangle$ of amplitude $\sim(1/N)(A/\sqrt{N}\omega)^2$ [see the discussion after Eq. (8)]. A specific case (fully polarized nuclear state) where the short-time expansion has only a very limited range of validity is discussed in Sec. IV. For the systems studied in experiment, we expect τ_x to be comparable to or longer than τ_e, since the experiments performed so far show time scales for the decay of $\langle h_z(t)\rangle$ on the order of seconds, minutes, or, in one case, even hours.[46] We note that it may be a demanding task to perform the fast and precise measurements required to obtain a Zeno effect in the present context. Still, experimental progress in the control of the nuclear field, such as that shown in Ref. 46, suggests that such measurements may be within reach in the near future.

We continue our discussion by deriving the effective Hamiltonian we use both for calculating short-time dynamics and the long-time behavior of $\langle h_z(t)\rangle$.

III. HAMILTONIAN

We aim to describe the dynamics of many nuclear spins surrounding a central-confined electron spin in a material with an s-type conduction band (e.g., GaAs, Si, etc.), where the dominant type of hyperfine interaction is the Fermi contact hyperfine interaction. The electron may be confined in many nanostructures such as nanowires, quantum dots, or defect centers. Under the assumption that other possible sources of nuclear-spin dynamics, such as nuclear quadrupolar splitting, are suppressed,[62] the two strongest interactions between nuclear spins in these nanostructures are the electron-mediated interaction ["J coupling" in NMR (Refs. 63 and 64)] and the direct dipole-dipole interaction. It turns out that, for a large number of nuclei N and up to magnetic fields of a few tesla (for GaAs), the contribution of the electron-mediated interaction to the initial decay of the Overhauser field is dominant (see Appendix A). The Hamiltonian contains three parts: the electron and nuclear Zeeman energies and the Fermi contact hyperfine interaction,

$$H = H_e + H_n + H_{en} = \epsilon_z S_z + \eta_z \sum_k I_k^z + \vec{S}\cdot\vec{h}. \tag{5}$$

Here, the operator

$$\vec{h} = \sum_k A_k \vec{I}_k \tag{6}$$

is the Overhauser field. Further, $\vec{S}$ is the electron spin and $\vec{I}_k$ is the nuclear spin at lattice site k that couples with strength $A_k = A\nu_0|\psi(r_k)|^2$ to the electron spin, where $A=\sum_k A_k$ is the total hyperfine coupling constant, ν_0 is the volume occupied by a single-nucleus unit cell, and $\psi(r_k)$ is the electron envelope wave function. We define the number of nuclear spins N interacting with the electron as the number of nuclear spins within an envelope-function Bohr radius of the confined electron.[27] The Bohr radius a_B for an isotropic electron envelope is defined through[27] $\psi(r_k)=\psi(0)e^{-(r_k/a_B)^q/2}$, where $q=1$ gives a hydrogenlike wave function and $q=2$ a Gaussian. Finally, ϵ_z and η_z are the electron and nuclear Zeeman splittings, respectively (we consider a homonuclear system). We derive an effective Hamiltonian for the electron-mediated interaction between nuclear spins, which is valid in a sufficiently large magnetic field. Using a standard Schrieffer-Wolff transformation[65] $H_{\mathrm{eff}}=e^S H e^{-S}$, to lowest order in H_{en}, with the transformation matrix $S=\sum_k A_k[(\epsilon_z+h_z-\eta_z+A_k/2)^{-1}S_+I_k^- -(\epsilon_z+h_z-\eta_z-A_k/2)^{-1}S_-I_k^+]/2$, which eliminates the off-diagonal terms between electron and nuclear spins, we find the effective Hamiltonian $H_{\mathrm{eff}}\simeq H_0+V$ (similar to Refs. 47, 49, and 66), where

$$H_0 = \epsilon_z S_z + \eta_z \sum_k I_k^z + S_z h_z, \tag{7}$$

$$V = \frac{1}{8}\sum_{kl} A_k A_l \left[\left(\frac{1}{2}+S_z\right)(B_k^+ I_k^- I_l^+ + I_l^- I_k^+ B_k^+) - \left(\frac{1}{2}-S_z\right)(B_k^- I_k^+ I_l^- + I_l^+ I_k^- B_k^-)\right]. \tag{8}$$

Here, $B_k^{\pm}=1/(\epsilon_z-\eta_z+h_z\pm A_k/2)$ and the raising and lowering operators are defined as $S_{\pm}=S_x\pm iS_y$ and similarly for $h_{\pm}$ and $I_k^{\pm}$. We note that H_{eff} neglects the transfer of spin polarization from the electron to the nuclei. The electron transfers an amount of angular momentum to the nuclear system on the order $(A/\sqrt{N}\omega)^2 \ll 1$ for $\omega \gg A/\sqrt{N}$.[27,67,68] Under the assumption that this amount is distributed equally among the N nuclear spins that interact appreciably with the electron, this results in a change in the average nuclear spin of order $\sim(1/N)(A/\sqrt{N}\omega)^2 \ll 1/N$. For $\omega\sim A$ these contributions are thus suppressed by a factor of $O(1/N)$ compared to the maximum decay of $\langle h_z(t)\rangle$ under H_{eff}, which is typically of $O(1/N)$ (see Secs. V and VI). For very special initial states where H_{eff} leads to no dynamics, e.g., for uniform polarization, the transfer of spin from the electron to the nuclei is the only source of nontrivial nuclear-spin dynamics and therefore should be taken into account. We discuss one such initial state, namely, a fully polarized nuclear system, in Sec. IV.

In the following we assume $I=1/2$ (Ref. 69) and neglect A_k in $B_k^{\pm}$, which are valid up to corrections suppressed by $A_k/(\epsilon_z-\eta_z+h_z)$. We further replace h_z in the denominator of $B_k^{\pm}$ by its initial expectation value $\langle h_z\rangle=\mathrm{Tr}\{h_z\rho(0)\}$ and introduce the effective electron Zeeman splitting

$$\omega = \epsilon_z - \eta_z + \langle h_z\rangle \approx \epsilon_z + \langle h_z\rangle. \tag{9}$$

This replacement assumes that the initial state does not change significantly and is valid up to corrections suppressed by σ/ω compared to the dynamics under H_{eff}. Here $\sigma=\sqrt{\langle h_z^2\rangle-\langle h_z\rangle^2}$ is the initial width of h_z. For an unpolarized equilibrium (infinite temperature) nuclear-spin state we have $\sigma\propto A/\sqrt{N}$ limiting the range of validity to $\omega\gg A/\sqrt{N}$,

$$V \cong \frac{1}{2\omega}\left[S_z\sum_{k\neq l} A_k A_l I_k^+ I_l^- + \frac{1}{2}\sum_k A_k^2 (S_z - I_k^z)\right], \tag{10}$$

where the terms $k=l$ are excluded in the sum over k and l. In the remainder of this paper, we will discuss the dynamics of the Overhauser field both at short and at long times in the regimes where a perturbative treatment in V is appropriate.

IV. SHORT-TIME EXPANSION

With respect to the Zeno effect as discussed in Sec. II, our main interest lies in the short-time behavior of the Overhauser-field z component $\langle h_z(t)\rangle$, where $\vec{h}$ is defined in Eq. (6). To calculate $\langle h_z\rangle_1$ and $\langle h_z\rangle_2$ [see Eq. (1)], we expand

$$\langle h_z(t)\rangle = \mathrm{Tr}\{h_z \exp(-iHt)\rho(0)\exp(iHt)\} \tag{11}$$

at short times. The first term $\langle h_z(0)\rangle = \mathrm{Tr}\{h_z\rho(0)\}$ gives the expectation value at time zero, while the t-linear term is proportional to $\langle h_z\rangle_1 = -i\,\mathrm{Tr}\{h_z[H,\rho(0)]\}$. Using the cyclicity of the trace we find that $\mathrm{Tr}\{h_z[H,\rho(0)]\} = \mathrm{Tr}\{[\rho(0),h_z]H\}$. Writing $\rho(0)=\rho_e(0)\otimes\rho_I(0)$ we have, for an initial nuclear-spin state $\rho_I(0)$ without transverse coherence, $[\rho_I(0),h_z]=0$ and thus the t-linear term vanishes. It might be possible to extend this result to more general randomly correlated initial nuclear-spin states, where terms involving transverse coherence or correlations are negligibly small due to a random phase.[70,71]

To determine the frequency of projective measurements required to induce a Zeno effect, we are interested in $\langle h_z\rangle_2 = -\mathrm{Tr}\{h_z[H,[H,\rho(0)]]\}$. We calculate $\langle h_z\rangle_2$ below using the effective Hamiltonian H_{eff} as derived in Sec. III. The range of validity is limited by higher-order terms in the effective Hamiltonian, which are proportional to $(h_+h_-)^n/\omega^{(n+1)}$ with $n=2,4,\ldots$. These higher-order terms give corrections to $\langle h_z\rangle_2$, which are suppressed by a factor of $(A/\sqrt{N}\omega)^n$. Thus the results for $\langle h_z(t)\rangle$ up to $O(t^2)$ given below are valid in the regime $\omega \gg A/\sqrt{N}$. Using $[h_z,H_0]=0$, we may simplify $\langle h_z\rangle_2$ considerably and we find for an arbitrary electron-spin state,

$$\langle h_z\rangle_2 = -\frac{1}{8\omega^2}\mathrm{Tr}_I\{h_z[\rho_I(0),h_+h_-]h_+h_-\}. \tag{12}$$

To further simplify, we assume a product initial state of the form

$$\rho(0) = \rho_e(0)\otimes\rho_I(0) = \rho_e(0)\otimes_k\rho_{I_k}, \tag{13}$$

$$\rho_{I_k} = 1/2 + f_k I_k^z; \quad f_k \equiv f_k(0) = 2\langle I_k^z(0)\rangle. \tag{14}$$

With this initial state we find

$$\langle h_z\rangle_2 = -\frac{1}{4\omega^2}\sum_{kl} f_k A_k^2 A_l^2\,\mathrm{Tr}_I\left\{h_z \underset{j\neq k,l}{\otimes}\left(\frac{1}{2}+f_j I_j^z\right)(I_k^z - I_l^z)\right\}. \tag{15}$$

Evaluating the commutators and the trace, we find for the decay of the Overhauser-field mean value $\langle h_z(t)\rangle$ up to corrections of $O(t^4)$,

$$\langle h_z(t)\rangle = \langle h_z(0)\rangle - \frac{t^2}{(8\omega)^2}\sum_{kl} A_k^2 A_l^2 (A_k - A_l)(f_k - f_l). \tag{16}$$

We note that both for uniform coupling constants $A_k = A/N$ and for uniform polarization $f_k = p, \forall k$, the t^2 term vanishes. This is, in fact, what one would expect, since H_{eff} only leads to a redistribution of polarization, and both for uniform polarization and uniform coupling constants, such a redistribution does not affect h_z. Rewriting the sum in Eq. (16) we obtain [again up to corrections of $O(t^4)$]

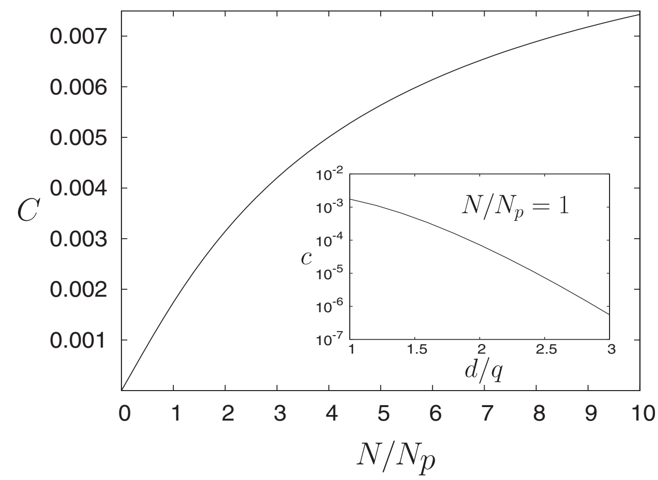

FIG. 2. Numerical prefactor c [given in Eq. (18)] of the t^2 term in the decay of the Overhauser-field mean value $\langle h_z(t)\rangle$. While c turns out to be independent (for $N \gtrsim 100$ in the case shown according to numerical summation) of the number N of nuclear spins within a Bohr radius of the electron envelope wave function, it does depend on the type of structure and the initial polarization. We show the case of a two-dimensional quantum dot with a Gaussian electron envelope ($d/q=1$). The dependence on the initial polarization is parametrized by N/N_p, where N_p is the number of nuclear spins that are polarized substantially (see text). Inset: Dependence of c on the ratio d/q for $N/N_p=1$. We see that, e.g., for a donor impurity with a hydrogenlike wave function ($d/q=3$) the prefactor c is more than three orders of magnitude smaller compared to the two-dimensional lateral quantum dot with $d/q=1$.

$$\frac{\langle h_z(t)\rangle}{\langle h_z(0)\rangle} = 1 - c\frac{t^2}{\tau_e^2}, \qquad \tau_e = \frac{N^{3/2}\omega}{A^2}, \tag{17}$$

with the numerical factor c only depending on the distribution of coupling constants through $\alpha_k = NA_k/A$ and the initial polarization distribution f_k through

$$c = \frac{1}{32Nc_0}\sum_{kl}\alpha_k^2\alpha_l^2(\alpha_k-\alpha_l)(f_k-f_l), \tag{18}$$

where $c_0 = \sum_k f_k\alpha_k$. We note that, up to the factor c (see Fig. 2), the time scale τ_e agrees with a previous rough estimate[60] for the time scale of nuclear-spin dynamics under the electron-mediated nuclear-spin interaction. In Table I we give τ_e for a variety of values of the number of nuclear spins N and of $\omega = \epsilon_z - \eta_z + \langle h_z\rangle$.

The coupling constants A_k have a different dependence on k, depending on the dimension d and the exponent q in the electron envelope wave function through[27] $A_k = A_0 e^{-(k/N)^{q/d}}$. For a donor impurity with a hydrogenlike exponential wave function we have $d=3$, $q=1$, and $d/q=3$, whereas for a two-dimensional quantum dot with a Gaussian envelope function we have $d=2$, $q=2$, and $d/q=1$. In Fig. 2 we show the constant c for the case $d/q=1$ and a particular choice of the polarization distribution. We give the dependence on d/q in the inset of Fig. 2. While c is independent of N for $N \gtrsim 100$, it changes considerably depending on the initial nuclear-spin state, which is parameterized by the f_k. Since

TABLE I. This table gives explicit values for the time scale τ_e of the t^2 term in the short-time expansion of $\langle h_z(t)\rangle$ [see Eq. (17)]. We give τ_e for various values of the number of nuclear spins N and of $\omega=\epsilon_z-\eta_z+\langle h_z\rangle$ for $I=1/2$ (Ref. 69). When $\omega=A/\sqrt{N}$ we are at the lower boundary of ω values for which the result for τ_e is valid. The parameters used are relevant for a lateral GaAs quantum dot: $A=90$ μeV and $g=-0.4$.

N	$A/g\mu_B\sqrt{N}$	τ_e at $\omega=A/\sqrt{N}$	τ_e at 100 mT	τ_e at 1 T	τ_e at 2 mT	τ_e at 5 T
1×10^3	49 mT	3 ns	6 ns	60 ns	119 ns	297 ns
1×10^4	16 mT	29 ns	188 ns	2 μs	4 μs	9 μs
1×10^5	4.9 mT	292 ns	6 μs	60 μs	119 μs	297 μs
1×10^6	1.6 mT	3 μs	188 μs	2 ms	4 ms	9 ms

there are neither experimental data nor theoretical calculations on the shape of the polarization distribution, we assume for the curves in Fig. 2 that it has the same shape as the distribution of coupling constants A_k, but with a different width reflected in the number of nuclear spins N_p that are appreciably polarized. The motivation for this choice is that if polarization is introduced into the nuclear-spin system via electron-nuclear-spin flip flops, the probability for these flip flops is expected to be proportional to some power of A_k/A_0. We denote the degree of polarization at the center by $p\in[-1,1]$. We may thus write $f_k=pe^{-(k/N_p)^{q/d}}$. We see in Fig. 2 that c grows monotonically with N/N_p, i.e., a localized polarization distribution ($N/N_p>1$) decays more quickly than a wide spread one ($N/N_p<1$).

In the context of state narrowing,[59–61] the short-time behavior of the width of the Overhauser field $\sigma(t)=\sqrt{\langle h_z^2(t)\rangle-\langle h_z(t)\rangle^2}$ is also of interest. Nuclear spin state narrowing, i.e., the reduction in σ, extends the electron-spin decoherence time. Repeating the above calculation for $\langle h_z^2(t)\rangle$ and using the result for $\langle h_z(t)\rangle$ we find [up to corrections of $O(t^4)$] for the variance of the Overhauser field,

$$\sigma^2(t)=\sigma^2(0)\left(1+c_\sigma\frac{t^2}{\tau_e^2}\right), \tag{19}$$

with the range of validity $\omega\gtrsim A/\sqrt{N}$, limited by higher-order corrections to the effective Hamiltonian as in the case of $\langle h_z(t)\rangle$. Here, the dimensionless constant c_σ is given by

$$c_\sigma=\frac{1}{16Nc_{\sigma0}}\sum_{kl}\alpha_k^2\alpha_l^2(\alpha_k-\alpha_l)(f_k-f_l)(f_k\alpha_k+f_l\alpha_l), \tag{20}$$

where $c_{\sigma0}=\sum_k\alpha_k^2(1-f_k^2)$. Taking the square root of $\sigma^2(t)$ and expanding it for $c_\sigma t^2/\tau_e^2\ll1$ we find for the width [up to corrections of $O(t^4)$],

$$\sigma(t)=\sigma(0)\left(1+c_\sigma\frac{t^2}{2\tau_e^2}\right). \tag{21}$$

Thus, also for the width of the Overhauser field, the initial dynamics is quadratic in time with the same dependence on A, N, and ω as the mean.

Special case: Full polarization

In this section we analyze the special case of a fully polarized nuclear-spin system where the effective Hamiltonian derived in Sec. III gives no dynamics, and thus the corrections due to the transfer of polarization from the electron to the nuclei become relevant. We thus must return to the full Hamiltonian in Eq. (5). Using the fact that the total spin $J_z=S_z+\sum_k I_k^z$ is a conserved quantity, we transform into a rotating frame where the Hamiltonian takes the form[27]

$$H'=(\tilde{\epsilon}_z+h_z)S_z+\frac{1}{2}(h_+S_-+h_-S_+), \tag{22}$$

with $\tilde{\epsilon}=\epsilon-\eta_z$. To have any dynamics for a fully polarized nuclear-spin system (all spins $|\uparrow\rangle$), the initial state of the electron must be $s_\Downarrow|\Downarrow\rangle+s_\Uparrow|\Uparrow\rangle$ with $s_\Downarrow\neq0$. Since the $|\Uparrow\rangle$ part gives no dynamics we consider $|\psi(0)\rangle=|\Downarrow;\uparrow\uparrow\ldots\uparrow\rangle$. At any later time we may thus write

$$|\psi(t)\rangle=a(t)|\psi(0)\rangle+\sum_k b_k(t)|\Uparrow;\uparrow\uparrow\ldots\uparrow\downarrow_k\uparrow\ldots\uparrow\rangle, \tag{23}$$

with $a(0)=1$ and $b_k(0)=0,\forall k$. The same case was studied in Refs. 67 and 68. However, this study was performed from the point of view of electron-spin decoherence. For the expectation value of $\langle h_z(t)\rangle$, we find, in terms of $a(t)$ and $b_k(t)$,

$$\langle h_z(t)\rangle=\langle\psi(t)|h_z|\psi(t)\rangle=\frac{A}{2}-\sum_k|b_k(t)|^2A_k, \tag{24}$$

where we have used the normalization condition $|a(t)|^2+\sum_k|b_k(t)|^2=1$. Using the time-dependent Schrödinger equation $i\partial_t|\psi(t)\rangle=H'|\psi(t)\rangle$, we obtain the differential equations for $a(t)$ and $b_k(t)$,

$$\dot{a}(t)=\frac{i}{4}(2\epsilon_z+A)a(t)-\frac{i}{2}\sum_k b_k(t)A_k, \tag{25}$$

$$\dot{b}_k(t)=-\frac{iA_k}{2}a(t)-\frac{i}{4}(2\epsilon_z+A-2A_k)b_k(t). \tag{26}$$

Inserting a power-series ansatz $a(t)=\sum_l a^{(l)}t^l$ and $b_k(t)=\sum_l b_k^{(l)}t^l$ into these equations and comparing coefficients yield recursion relations of the form

$$a^{(l+1)}=\frac{i}{4(l+1)}(2\epsilon_z+A)a^{(l)}-\frac{i}{2(l+1)}\sum_k b_k^{(l)}A_k, \tag{27}$$

$$b_k^{(l+1)} = -\frac{iA_k}{2(l+1)}a^{(l)} - \frac{i}{4(l+1)}(2\epsilon_z + A - 2A_k)b_k^{(l)}. \quad (28)$$

Iterating these recursion relations using $a(0)=1$ and $b_k(0)=0, \forall k$, we find, neglecting corrections of $O(t^4)$,

$$\frac{\langle h_z(t)\rangle}{\langle h_z(0)\rangle} = 1 - \frac{1}{2A}\sum_k A_k^3 t^2. \quad (29)$$

For the case of a two-dimensional quantum dot with Gaussian envelope wave function where we have $A_k = Ae^{-k/N}/N$, we find, evaluating $\Sigma_k A_k^3$ by turning it into an integral in the continuum limit $N \gg 1$ [again up to corrections of $O(t^4)$],

$$\frac{\langle h_z(t)\rangle}{\langle h_z(0)\rangle} = 1 - \frac{1}{6}\left(\frac{t}{\tau_c}\right)^2, \quad (30)$$

where $\tau_c = N/A$. To obtain the range of validity for this result we go to higher order in t. Again for the case of a two-dimensional quantum dot with Gaussian envelope wave function we find up to $O(t^4)$, neglecting terms that are suppressed by $O(1/N)$ in the t^4 term,

$$\frac{\langle h_z(t)\rangle}{\langle h_z(0)\rangle} = 1 - \frac{1}{6}\left(\frac{t}{\tau_c}\right)^2 + \frac{1}{18}\left(\frac{t}{\tau_4}\right)^4. \quad (31)$$

Here, $\tau_4 = 2\sqrt{N}/\sqrt{A(2\epsilon_z + A)}$. This shows that in some cases the higher-order terms in the short-time expansion can have a considerably shorter time scale. Comparing the short-time expansion with a calculation for $\langle S_z\rangle$ in the case of uniform coupling constants[67] suggests that the full dynamics contain oscillations with frequency $\propto \epsilon_z + A/2$, thus, limiting the range of validity of the short-time expansion to $t \ll (\epsilon_z + A/2)^{-1}$.

To finish our discussion on the short-time dynamics and on the Zeno effect, we point out that the main result of this section, namely, the time scale τ_e and the constant c [Eqs. (17) and (18)] for the quadratic term in the short-time expansion of $\langle h_z(t)\rangle$ is what sets the condition on the repetition rate τ_m as discussed in Sec. II [see Eq. (4)]. With this we move on to the study of the long-time behavior. We first show the results of a Dyson-series expansion in Sec. V and in Sec. VI we treat the problem using the generalized master equation (GME) showing that the Dyson-series expansion gives the leading-order contribution in A/ω.

V. DYSON-SERIES EXPANSION

In this section we calculate the expectation value of the Overhauser field $\langle h_z(t)\rangle$ in a Dyson-series expansion up to second order in the interaction V. This allows us to obtain the full time dynamics of $\langle h_z(t)\rangle$. Since the Dyson-series expansion is not a controlled expansion (it leads to secular divergences in time at higher order), we will only see from the generalized master-equation calculation in Sec. VI that the Dyson-series result gives the correct leading-order contribution in A/ω. Thus, the results in this section are expected to be valid in the regime $\omega \gg A$. The results in this section can also be obtained from the generalized master-equation approach presented in Sec. VI. However, the Dyson-series calculation is more accessible.

We transform all operators into the interaction picture by $\widetilde{\mathcal{O}} = e^{iH_0t}\mathcal{O}e^{-iH_0t}$. In the interaction picture we have $\langle h_z(t)\rangle = \mathrm{Tr}\{\widetilde{h}_z\widetilde{\rho}(t)\}$ with $\widetilde{h}_z = h_z$ since $[H_0, h_z]=0$. Expanding $\widetilde{\rho}(t)$ in a Dyson series we find[72]

$$\widetilde{\rho}(t) = \rho(0) - i\int_0^t dt'[\widetilde{V}(t'),\rho(0)] - \int_0^t dt'\int_0^{t'} dt''[\widetilde{V}(t'),[\widetilde{V}(t''),\rho(0)]] + O(\widetilde{V}^3), \quad (32)$$

where

$$\widetilde{V}(t) \equiv e^{iH_0t}Ve^{-iH_0t} = \frac{S_z}{2\omega}\sum_{k\neq l} e^{iS_z(A_k - A_l)t}I_k^+ I_l^-. \quad (33)$$

We assume again the same initial state as in Sec. IV and thus the term that is linear in $\widetilde{V}$ will drop out under the trace as it only contains off-diagonal terms. From the remaining two terms we find

$$\langle h_z(t)\rangle = \langle h_z(0)\rangle + \frac{1}{8\omega^2}\sum_{k\neq l}\frac{A_k^2 A_l^2(f_k - f_l)}{A_k - A_l} \times \left\{\cos\left[(A_k - A_l)\frac{t}{2}\right] - 1\right\}. \quad (34)$$

We first verify that this result is consistent with the short-time expansion in Sec. IV. For this we use $A_k \leq A_0 \propto A/N$ and thus for times $t \ll \tau_c = N/A$ we may expand the cosine in the above expression recovering (to second order in t) the result in Eq. (16). For the full time dynamics we note that the sum over cosines leads to a decay on a time scale of $\tau_c = N/A$ since for $t > \tau_c$ the different cosines interfere destructively. We illustrate this with an example: For a particular choice of the initial polarization distribution ($d/q=1$ and $N_p = N$) we may evaluate the sum in Eq. (34) in the continuum limit and find

$$\frac{\langle h_z(t)\rangle}{\langle h_z(0)\rangle} = 1 - \frac{p}{8N}\frac{A^2}{\omega^2}g(t/\tau_c). \quad (35)$$

The function $g(t)$ is explicitly given by

$$g(t) = \frac{1}{t^4}\left[t^4 - 16t^2 + 64t\sin\left(\frac{t}{2}\right) - 256\sin^2\left(\frac{t}{4}\right)\right], \quad (36)$$

with $g(0)=0$ and $g(t\to\infty)=1$. We thus find a power-law decay on a time scale τ_c by an amount of $O(1/N)$. Since the sum of cosines in Eq. (34) decays, the remaining time-independent sum gives the stationary value (up to the Poincaré recurrence time[73])

$$\frac{\langle h_z\rangle_{\mathrm{stat}}}{\langle h_z(0)\rangle} = 1 - \left(\frac{A}{\omega}\right)^2\frac{1}{4N^2c_0}\sum_{k\neq l}\frac{\alpha_k^2\alpha_l^2(f_k - f_l)}{\alpha_k - \alpha_l}. \quad (37)$$

For a system with a large number of nuclear spins $N \gg 1$ and a sufficiently smooth polarization distribution, this stationary value differs only by a term of $O(1/N)$ from the initial value, i.e., $\langle h_z\rangle_{\mathrm{stat}}/\langle h_z(0)\rangle = 1 - O(1/N)$. This can be seen in Fig. 3

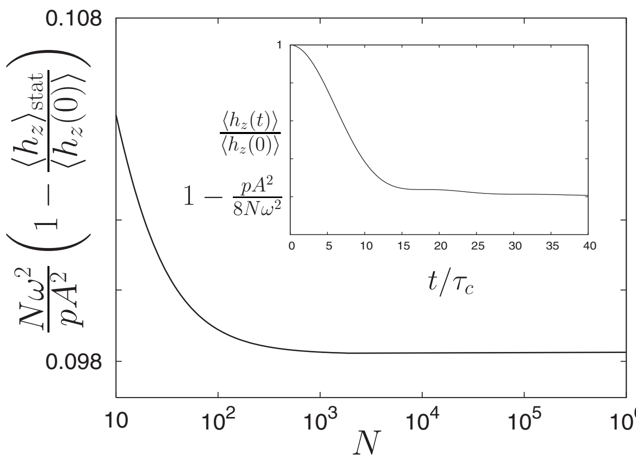

FIG. 3. In this figure we show the N dependence of $1-\langle h_z\rangle_{\rm stat}/\langle h_z(0)\rangle$ [see Eq. (37)], i.e., the part by which $\langle h_z\rangle$ decays in units of $pA^2/N\omega^2$, in the regime $\omega \gg A$. This plot is for a three-dimensional defect center with a hydrogenlike electron envelope ($d/q=3$) and the initial polarization is parameterized by $N/N_p=0.5$ as described in Sec. IV. On this choice of polarization distribution the decay is $O(1/N)$. The inset shows the full time dynamics of $\langle h_z(t)\rangle/\langle h_z(0)\rangle$ as given in Eq. (35) for $d/q=1$, $N\gg 1$, and $N/N_p=1$. We see that the decay occurs on a time scale of $\tau_c=N/A$.

where we show the N dependence of $1-\langle h_z\rangle_{\rm stat}/\langle h_z(0)\rangle$, i.e., the part by which $\langle h_z\rangle$ decays. The parameters in Fig. 3 are taken for a three-dimensional defect center with a hydrogen-like electron envelope ($d/q=3$) and the initial polarization $N/N_p=0.5$ as described in Sec. IV. For this choice of polarization distribution the decay is $O(1/N)$. We also find a $O(1/N)$ behavior for other values of the parameters d/q and N/N_p and thus expect this to be generally true for a smoothly varying initial polarization distribution. The inset of Fig. 3 shows the full-time dynamics of $\langle h_z(t)\rangle$ as given in Eq. (35) for $d/q=1$, $N\gg 1$, and $N/N_p=1$.

We note that the fourth order of a Dyson-series expansion gives secular terms (diverging in t). We thus move on to treat the long-time behavior using a master-equation approach, which avoids these secular terms and shows that the Dyson-series result gives the correct leading-order term in A/ω.

VI. GENERALIZED MASTER EQUATION

In this section we study the decay of the Overhauser-field mean value $\langle h_z(t)\rangle$ using the Nakajima-Zwanzig GME in a Born approximation. The results in this section are valid in the regime $\omega\gg A$ since higher-order corrections to the Born approximation are suppressed by a factor of $(A/\omega)^2$.

We start from the GME,[73] which for $P_k\rho(0)=\rho(0)$ reads

$$P_k\dot{\rho}(t) = -iP_kLP_k\rho(t) - \int_0^t dt' P_kLe^{-iQL(t-t')}QLP_k\rho(t'), \tag{38}$$

where $L=L_0+L_V$ is the Liouville superoperator defined as $(L_0+L_V)\mathcal{O}=[H_0+V,\mathcal{O}]$. The projection superoperator P_k must preserve $\langle I_k^z(t)\rangle$ and we choose it to have the form $P_k=\rho_e(0)\mathrm{Tr}_e\otimes P_{dk}\otimes_{l\neq k}\rho_{I_l}(0)\mathrm{Tr}_{I_l}$, where P_{dk} projects onto the diagonal in the subspace of nuclear spin k and is defined as $P_{dk}\mathcal{O}=\Sigma_{s_k=\uparrow,\downarrow}|s_k\rangle\langle s_k|\langle s_k|\mathcal{O}|s_k\rangle$. Further, $Q=1-P_k$. In a standard Born approximation and using the same initial conditions as above, i.e., a product state and no transverse coherence in the nuclear-spin system, we obtain the following integrodifferential equation for $\langle I_k^z(t)\rangle$:

$$\langle \dot{I}_k^z(t)\rangle = -\frac{A_k^2}{8\omega^2}\int_0^t d\tau \sum_{l,l\neq k} A_l^2\cos\left[\frac{\tau}{2}(A_k-A_l)\right] \times[\langle I_k^z(t-\tau)\rangle - \langle I_l^z(0)\rangle]. \tag{39}$$

The Born approximation goes to order of L_V^2 in the expansion of the self-energy. Higher-order corrections in L_V are estimated to give contributions to the right-hand side of Eq. (39) that are suppressed by a factor of $(A/\omega)^2$. We expect the results of this section to be valid at least for $\omega\gg A$, although it could in principle happen that [as in the case of the decay of $\langle S_z(t)\rangle$ (Ref. 27)] the result for the stationary value has a larger regime of validity. On the other hand, it cannot be generally excluded that higher-order contributions could dominate at sufficiently long times. Integrating Eq. (39) we find the formal solution

$$\langle I_k^z(t)\rangle = \langle I_k^z(0)\rangle - \frac{A^2}{\omega^2}\frac{\alpha_k^2}{8}\int_0^t dt'\int_0^{t'} d\tau\sum_{l,l\neq k}A_l^2\cos\left[\frac{\tau}{2}(A_k-A_l)\right] \times[\langle I_k^z(t'-\tau)\rangle-\langle I_l^z(0)\rangle]. \tag{40}$$

This shows that $\langle I_k^z(t)\rangle=\langle I_k^z(0)\rangle+O[(A/\omega)^2]$ and we may thus iterate this equation and replace $\langle I_k^z(t'-\tau)\rangle$ in the integral by $\langle I_k^z(0)\rangle$. This implies, up to corrections of $O[(A/\omega)^4]$,

$$\langle I_k^z(t)\rangle = \langle I_k^z(0)\rangle - \frac{A_k^2}{16\omega^2}\sum_{l,l\neq k}A_l^2(f_k-f_l)\int_0^t dt'\int_0^{t'} d\tau \times\cos\left[\frac{\tau}{2}(A_k-A_l)\right]. \tag{41}$$

Performing the integrals and summing over the $\langle I_k^z(t)\rangle$ weighted by their coupling constants A_k, we recover the Dyson-series result in Eq. (34). This shows that the Dyson-series expansion gives the leading-order contribution in A/ω.

For the analytical solution of Eq. (39) in the stationary limit we perform a Laplace transformation, solve the resulting equation in Laplace space, and calculate the residue of the pole at $s=0$, which yields (up to the recurrence time)

$$\langle I_k^z\rangle_{\rm stat} = \lim_{T\to\infty}\frac{1}{T}\int_0^T\langle I_k^z(t)\rangle dt = \lim_{s\to 0} s\langle I_k^z(s)\rangle = \frac{1}{Z_k}\sum_l P_k(l)\langle I_l^z(t=0)\rangle, \tag{42}$$

with $Z_k=\Sigma_l P_k(l)$. We see that $\langle I_k^z\rangle_{\rm stat}$ is determined by weighting the neighboring $\langle I_l^z(t=0)\rangle$ with the probability distribution $P_k(l)/Z_k$, which is explicitly given by

$$P_k(l) = \begin{cases} A_l^2/(A_k - A_l)^2: & l \neq k, \\ 2\omega^2/A_k^2: & l = k. \end{cases} \quad (43)$$

We point out that $\langle I_k^z \rangle_{\text{stat}}$ can be either smaller or larger than $\langle I_l^z(t=0) \rangle$ and that $\Sigma_k \langle I_k^z \rangle_{\text{stat}} = \Sigma_k \langle I_l^z(t=0) \rangle$ since the total spin is a conserved quantity. Again expanding the result in Eq. (42) to leading order in A/ω and summing over the nuclear spins weighted by their coupling constants A_k, we recover the same result found in the Dyson-series calculation in Eq. (37). Intuitively one would expect a decay even at high fields (although a very slow one) to a state with uniform polarization. The fact that our calculation shows no such decay suggests that the Knight-field gradient, i.e., the gradient in the additional effective magnetic field seen by the nuclei, due to the presence of the electron, is strong enough to suppress such decay if the flip-flop terms are sufficiently suppressed. Applying a large magnetic field thus seems to be an efficient strategy to prevent the Overhauser field from decaying.

As a side remark, we would like to point out that in this regime of only partial decay, repeated measurements on the Overhauser field can actually enhance the decay of $\langle h_z(t) \rangle$. This occurs when the measurements are performed at intervals longer than the time scale for decay to the stationary value ($\tau_c = N/A$ as discussed in Sec. V). Performing a projective measurement at a time $t > \tau_c$ resets the initial condition and thus again a small decay occurs. Repeating these measurements at intervals longer than τ_c thus allows for a decay of $\langle h_z(t) \rangle$ to zero.

VII. CONCLUSION

We have studied the dynamics of the Overhauser field generated by the nuclear spins surrounding a bound electron. We focused our analysis on the effect of the electron-mediated interaction between nuclei due to the hyperfine interaction. At short times we find a quadratic initial decay of the Overhauser-field mean value $\langle h_z(t) \rangle$ on a time scale $\tau_e = N^{3/2}\omega/A^2$. Performing repeated strong measurements on h_z leads to a Zeno effect with the decay changing from quadratic to linear with a time scale that is prolonged by a factor of τ_e/τ_m, where τ_m is the time between consecutive measurements. In Secs. V and VI we have addressed the long-time decay of $\langle h_z(t) \rangle$ using a Dyson-series expansion and a generalized master-equation approach. Both show that $\langle h_z(t) \rangle$ only decays by a fraction of $O(1/N)$ for a sufficiently smooth polarization distribution and large magnetic field.

Overall, the strategy to preserve the Overhauser field contains two tools. The first is to apply a strong external magnetic field ($\omega \gg A$), which limits the decay to a fraction of $O(1/N)$. In case a strong magnetic field is not desirable or achievable, the second tool is to make use of the Zeno effect and perform repeated projective measurements on the Overhauser field leading to a slow down of the decay.

It remains a subject of further study beyond the scope of this work whether, and on what time scale, the combination of electron-mediated interaction and direct dipole-dipole interaction may lead to a full decay of the Overhauser field. Another interesting question concerns the distribution of nuclear polarization within a quantum dot or defect center and its dependence on the method that is used to polarize the system.

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APPENDIX A: ESTIMATION OF DIPOLE-DIPOLE CONTRIBUTION

In this appendix we estimate the time scale arising from the direct secular (terms conserving $I_{z,\text{tot}} = \Sigma_k I_k^z$) dipole-dipole interaction in the short-time expansion of the Overhauser-field mean value $\langle h_z(t) \rangle$. This gives us the range of validity of our calculation in the main text that only took into account the electron-mediated interaction between nuclei. Let us thus consider the situation where the external magnetic field is very high such that the electron-mediated flip-flop terms are fully suppressed. In this case the Hamiltonian has the form $H_{dd} = H_{0,dd} + V_{dd}$ with

$$H_{0,dd} = \epsilon_z S_z + \eta_z \sum_k I_k^z + S_z h_z - 2 \sum_{k \neq l} b_{kl} I_k^z I_l^z, \quad (A1)$$

$$V_{dd} = \sum_{k \neq l} b_{kl} I_k^+ I_l^- . \quad (A2)$$

Here, $b_{kl} = \gamma_I^2 [3\cos^2(\theta_{kl}) - 1]/4r_{kl}^3$ with θ_{kl} being the angle between a vector from nucleus k to nucleus l and the z axis, and r_{kl} being the distance between the two nuclei.[64] Further, γ_I is the nuclear gyromagnetic ratio. For the short-time expansion, only the off-diagonal terms are relevant, since $[h_z, H_0] = [\rho(0), H_0] = 0$. These off-diagonal terms in the case of the electron-mediated interaction are $S_z \Sigma_{k \neq l} A_k A_l I_k^+ I_l^- / 2\omega$ [see Eq. (10)]. Replacing $A_k A_l / 2\omega$ with b_{kl} in the result for the short-time expansion in Eq. (16) and also taking into account the factor of 1/4 that comes from S_z^2 in the electron-mediated case we find

$$\langle h_z(t) \rangle_{\text{dip-dip}} = \langle h_z(0) \rangle - \frac{t^2}{4} \sum_{kl} b_{kl}^2 (A_k - A_l)(f_k - f_l). \quad (A3)$$

To estimate, we restrict the sum to nearest neighbors as b_{kl} falls off with the third power of the distance between the two nuclei. Assuming $f_k = (A_k/A_0)^{N/N_p}$ we find up to corrections of $O(t^4)$,

$$\frac{\langle h_z(t) \rangle_{\text{dip-dip}}}{\langle h_z(0) \rangle} \approx 1 - \frac{t^2}{\tau_d^2}, \quad \tau_d = \frac{\sqrt{N_p N}}{b}, \quad (A4)$$

with b being the nearest-neighbor dipole-dipole coupling. For GaAs we have $b \sim 10^3\ \text{s}^{-1}$ [with $\gamma_I / 2\pi \approx 10$ MHz/T (Ref. 74)]. For $NN_p \gg 1$ we have $\tau_d \gg 10^{-3}$ s.[69] In the magnetic-field range shown in Table I we thus have τ_d

$\gg \tau_e/\sqrt{c}$, which justifies neglecting the direct dipole-dipole coupling in the short-time expansion.

APPENDIX B: MEASUREMENT ACCURACY

The description of the Zeno effect in Sec. II relied on the assumption that the measurements on h_z set all off-diagonal elements of the density matrix to zero. This assumption requires, on one hand, a perfect measurement accuracy for h_z (we discuss deviations from that below), but on the other hand it also requires the h_z eigenstates to be nondegenerate. For nondegenerate h_z eigenstates a measurement of h_z fully determines the polarization distribution f_k and we may thus write ρ_I after the measurement again as a direct product with $\rho_{I_k}(\tau_m)=1/2+f_k(\tau_m)I_k^z$. After the measurement, we thus again have the same time evolution for $\langle h_z(t)\rangle$ as given in Eq. (16), but with f_k replaced by $f_k(\tau_m)$. Iterating Eq. (16) for the case of m consecutive measurements at intervals τ_m one obtains Eq. (3).

Instead of the idealized assumption of a projective measurement we now allow imperfect measurements. To describe these measurements we use a so-called positive operator valued measure (POVM).[54] In a general POVM measurement the density matrix changes according to[54]

$$\rho \to \rho' = \int \sqrt{F_y}\rho\sqrt{F_y}dy, \tag{B1}$$

when averaging over all possible measurement outcomes y. The probability to measure outcome y is given by $P(y)=\mathrm{Tr}\{\rho F_y\}$ and the condition $\int dy F_y=1$ ensures that the probabilities sum to unity. We consider the nuclear density matrix ρ_I in a basis of h_z eigenstates $|n\rangle$ with $h_z|n\rangle=h_z^n|n\rangle$. We denote the matrix elements of ρ_I by $\rho_I(n,m)=\langle n|\rho_I|m\rangle$. For the following description we assume that the diagonal of the nuclear-spin density matrix before the measurement is Gaussian distributed around its mean value $\langle h_z\rangle$ with a width σ, i.e.,

$$\rho_I(n,n) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(h_z^n-\langle h_z\rangle)^2}{2\sigma^2}\right]. \tag{B2}$$

For an unpolarized equilibrium (infinite temperature) state, the width is $\sigma \propto A/\sqrt{N}$. Here, σ can take any value. We consider a measurement with a Gaussian line shape of width η. Since we aim to describe measurements that at least partially project the nuclear-spin state, we have $\eta \ll \sigma$. The POVM that describes such a measurement is given by

$$F_y = \sum_n f(n,y)|n\rangle\langle n|, \tag{B3}$$

with

$$f(n,y) = \frac{1}{\sqrt{2\pi}\eta}\exp\left[-\frac{(h_z^n-\langle h_z\rangle-y)^2}{2\eta^2}\right]. \tag{B4}$$

It is straightforward to calculate the probability $P(y)$ for obtaining the measurement result $\langle h_z\rangle+y$,

$$P(y) = \frac{1}{\sqrt{2\pi(\sigma^2+\eta^2)}}\exp\left[-\frac{y^2}{2(\sigma^2+\eta^2)}\right]. \tag{B5}$$

Clearly, the probabilities add up to one $[\int P(y)dy=1]$ as they should. For the diagonal of the nuclear-spin density matrix after a measurement with outcome $\langle h_z\rangle+y$ we find

$$\rho_I'(n,n;y) = \frac{\rho_I(n,n)f(n,y)}{P(y)} \overset{\eta\ll\sigma}{=} f(n,y). \tag{B6}$$

Also, when weighting the $\rho_I'(n,n;y)$ with their probabilities for occurring, we find $\int \rho_I'(n,n;y)P(y)dy=\rho_I(n,n)$. Using Eq. (B1) we thus find for the matrix elements after a measurement, when averaging over all possible measurement outcomes,

$$\rho_I(n,m) \to \rho_I'(n,m) = \rho_I(n,m)\int \sqrt{f(n,y)f(m,y)}dy, \tag{B7}$$

which leads to

$$\rho_I'(n,m) = \rho_I(n,m)\exp\left[-\frac{(h_z^n-h_z^m)^2}{8\eta^2}\right]. \tag{B8}$$

To reduce the off-diagonal elements, the measurement accuracy must be better than the difference in eigenvalues. In the limit $\eta\to 0$ a projective measurement is recovered, which sets all off-diagonal elements to zero. Up to t^2 in the short-time expansion, only off-diagonal elements between states that differ at most by two flip flops can become nonzero. Thus, to have at least a partial Zeno effect[55] resulting from the off-diagonal elements being partially reduced, the requirement on the measurement accuracy is $\eta \lesssim |h_z^n-h_z^m|$ with $|n\rangle=I_k^+I_l^-I_p^+I_q^-|m\rangle$. For coupling constants $A_k=Ae^{-k/N}/N$, we have typically $h_z^n-h_z^m\sim A/N$. Besides suppressing the off-diagonal elements of ρ_I through a measurement, there are also "natural" decoherence mechanisms, such as inhomogeneous quadrupolar splittings, electron-phonon coupling, or spin-lattice relaxation that can lead to a reduction in the off-diagonal elements of ρ_I.

As mentioned earlier, there are several proposals[59–61] to implement a projective measurement of h_z. All of these proposed techniques rely on the fact that the dynamics of the electron spins confined in the dots depends on the Overhauser field. Thus, a measurement of the electron-spin dynamics allows one to indirectly measure h_z. The proposal in Ref. 59 is designed for optically active self-assembled quantum dots and makes use of an h_z-dependent frequency shift in an exciton transition. Numerical calculations for this method[59] show that an increase in the electron-spin coherence time by a factor of 100 is achievable with a preparation time of 10 μs, which corresponds to a measurement accuracy of $\eta=\sigma_0/100=A/100\sqrt{N}$. The proposal in Ref. 60 considers gate-defined double quantum dots, such as the ones in Refs. 11, 34, 75, and 76. The measurements of electron-spin dynamics are achieved through spin-to-charge conversion and detection of the charge by a nearby quantum point contact (QPC). An experimental recipe in the context of the setup in Ref. 11 is presented in Ref. 77. The narrowing (measurement accuracy) achievable with this method essentially relies on a good time resolution (better than the time scale for electron-spin dynamics) of the QPC charge detection, which by now has reached a few hundred nanoseconds.[78]

An alternative read-out scheme is to use a double dot in the spin-blockade regime,[13] where one spin, say the one in the left dot, is manipulated, while the one in the right dot is only needed for the readout. The system is initialized to the triplet $|\uparrow\uparrow\rangle$, which is spin blocked since the triplet $T(0,2)$ with two electrons on the right dot is energetically not accessible. The $|\downarrow\uparrow\rangle$ state, however, may tunnel to the $S(0,2)$ singlet and from there one electron can tunnel to the right lead if energetically allowed, leaving the two-electron system in a (0,1) state. The efficiency of this read-out scheme generally depends on a large separation of the transition rates for the $|\uparrow\rangle$ and the $|\downarrow\rangle$ states. If such a large separation, i.e., a good spin blockade, can be achieved, this read-out method offers the potential for rapid consecutive electron-spin measurements and thus also for accurate measurements of the Overhauser field. A variant of this scheme is to adiabatically pulse the double dot between different two-electron-spin states and to use a nearby QPC for readout (see, e.g., Refs. 26 and 79). A detailed analysis of using this read-out scheme for projective measurements of h_z has, to our knowledge, not been undertaken yet and is beyond the scope of this work.

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